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1983 J. Phys. A: Math. Gen. 16 271

(http://iopscience.iop.org/0305-4470/16/2/010)

J. Phys. A: Math. Gen. 16 (1983) 271–277. Printed in Great Britain

On some exact solutions of the nonlinear Dirac equation

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Received 1 April 1982, in final form 19 July 1982

Abstract. Multiparametrical exact solutions of the nonlinear Dirac equation are found within the framework of the group-theoretical approach. A procedure for generating new solutions from known ones is presented. The solutions obtained are analytic in the coupling constant, vanishing at infinity and describe the oscillations with the corresponding solutions of the equation without self-interaction as amplitude.

1. Introduction

In this paper the multiparametrical exact solutions of nonlinear Dirac equations are obtained with the help of the group-theoretical approach. The equations have the form:

$$[\mathrm{i}\gamma_\mu\,\partial/\partial x_\mu - m - \lambda(\bar{\psi}(x)\psi(x))^k]\psi(x) = 0 \tag{1}$$

$$[\mathrm{i}\gamma_\mu\,\partial/\partial x_\mu - \lambda(\bar{\psi}(x)\psi(x))^k]\psi(x) = 0 \tag{2}$$

where γ_μ are 4×4 Dirac matrices (see, for example, Bjorken and Drell 1964) $\mu, \nu \ldots = 0, 1, 2, 3$; $m \neq 0$, k, λ are arbitrary real constants. We use the summation convention for repeated indices.

It is worthwhile to distinguish equations (1) with $m \neq 0$ from (2) because of their considerable different symmetry properties.

In order to find the exact solutions, we exploit the fact that equation (1) is invariant under the Poincaré group P(1, 3), equation (2) is invariant under the Weyl group W(1, 3) = {P(1, 3), D} when $k \neq \frac{1}{3}$ and under the conformal group C(1, 3) when $k = \frac{1}{3}$. We also show how to draw new families of solutions from known ones.

Fushchich (1981) has obtained multiparametrical exact solutions of many-dimensional nonlinear scalar sine-Gordon, Liouville, Hamilton–Jacobi, eikonal, Born–Infeld (Fushchich and Serow 1982), Schrödinger (Fushchich and Moskaliuk 1981) equations by the method recently proposed (Fushchich 1981). Here we slightly generalise this method to fit it for the system of partial differential equations.

2. The method

Let Q be an infinitesimal operator of local transformations admitted by equations (1) or (2). The general form of such an operator is

$$Q = \xi^\mu(x)\,\partial_\mu + \eta(x) \tag{3}$$

0305-4470/83/020271+07$02.25

where $\xi^\mu(x)$ are scalar functions of x, $\eta(x)$ denotes the 4×4 matrix depending on x and $\partial_\mu \equiv \partial/\partial x_\mu$.

The operator Q gives the possibility to find exact solutions of equations (1) or (2) in such a manner.

For the solutions to be found we adopt the *ansatz* suggested by Fushchich (1981)

$$\psi(x) = A(x)\varphi(\omega) \tag{4}$$

where the nonsingular 4×4 matrix $A(x)$ can be defined from the equation

$$QA(x) \equiv (\xi^\mu(x)\,\partial_\mu + \eta(x))A(x) = 0, \tag{5}$$

ω are invariants of the differential part of the operator Q, i.e. functions satisfying

$$\xi^\mu(\partial\omega/\partial x_\mu) = 0 \tag{6}$$

or the equivalent Lagrange–Euler system

$$\frac{dx_0}{\xi^0(x)} = \frac{dx_1}{\xi^1(x)} = \frac{dx_2}{\xi^2(x)} = \frac{dx_3}{\xi^3(x)} \overset{\text{def}}{=} d\tau. \tag{7}$$

$\varphi(\omega)$ is the new unknown four-component spinor field depending on new variables, ω, the number of which is one less than the number of variables x.

When $A(x)$ and ω are known, then the substitution of expression $A(x)\varphi(\omega)$ in place of $\psi(x)$ in equations (1) and (2) leads to a system of differential equations for $\varphi(\omega)$ which is often rather easy to solve.

Another procedure for determining the *ansatz* (4) explicitly is to solve, besides equation (7), the following system of ordinary differential equations

$$d\psi/d\tau = -\eta(x(\tau))\psi. \tag{8}$$

If we insert in the general solution of this system, the value τ defined from (7), and consider constants of integration as functions of ω then we shall obtain the *ansatz* (4) possessing the properties (5) and (6). Let us discuss the procedure of generating new solutions from known ones.

The general form of transformations generated by operator Q (3) is

$$x \to x' = f(x, \theta), \qquad \psi(x) \to \psi'(x') = R(x, \theta)\psi(x) \tag{9}$$

where $R(x, \theta)$ is a 4×4 matrix, θ is a parameter of transformations. Formula (9) implies that

$$\psi_{\text{new}}(x) = R^{-1}(x, \theta)\psi_{\text{old}}(x') \tag{10}$$

will be a solution of the equation which admits operator Q as well as $\psi_{\text{old}}(x)$ ($R^{-1}(x, \theta)$ denotes the inverse matrix).

Remark. Equation (5) is the consequence of the following obvious condition: the solutions having the form (4) do not produce new solutions by virtue of the procedure stated above when transformations (9) are generated by the same operator Q (3). Indeed, we have according to (4) and (10)

$$R^{-1}(x, \theta)A(x')\varphi(\omega') = A(x)\varphi(\omega). \tag{11}$$

$\omega' = \omega$ because ω are invariants of the operator Q:

$$A(x') = A(x) + \theta\xi^\mu(x)(\partial A(x)/\partial x_\mu) + \dots \tag{12}$$

$$R(x,\theta)=I-\theta\eta(x)+\ldots \tag{13}$$

If we substitute (12) and (13) into (11) and retain terms linear in θ then we obtain (5).
It is clear now that it is the form of transformations (9) that leads to the *ansatz* (4).

3. The solutions

First of all we give an example of a conformally invariant solution to the Dirac equation (2) with Gursey (1956) nonlinearity $k=\frac{1}{3}$ which ensures the conformal invariance of the equation. This solution has the form

$$\psi(x)=\frac{\gamma x}{(x_\nu x^\nu)^2}\exp[\mathrm{i}\lambda\varkappa(\gamma\beta)\omega]\chi \equiv \frac{\gamma x}{(x_\nu x^\nu)^2}\left(\cos(\lambda\varkappa\beta\omega)+\mathrm{i}\frac{\gamma\beta}{\beta}\sin(\lambda\varkappa\beta\omega)\right)\chi \tag{14}$$

where $\omega=\beta x/x_\nu x^\nu$, β^ν are arbitrary real constants, $\beta\equiv(\beta_\nu\beta^\nu)^{1/2}>0$. χ denotes a space–time independent spinor,

$$\bar{\chi}\chi=a=\text{constant}, \qquad \varkappa=a^{1/3}/\beta^\nu\beta_\nu \qquad \beta x\equiv\beta^\nu x_\nu,\ \text{etc.}$$

The solution (14) was sought for in the form

$$\psi(x)=[\gamma x/(x_\nu x^\nu)^2]\varphi(\omega) \qquad \omega=\beta x/x_\nu x^\nu \tag{15}$$

obtained with the help of the conformal transformation operator

$$Q_{\text{conf}}=c^\mu k_\mu=2(cx)x\,\partial-x^2c\,\partial+(\gamma c\gamma x+2cx) \tag{16}$$

$$Q_{\text{conf}}[\gamma x/(x_\nu x^\nu)^2]\equiv 0 \qquad [2(cx)x\,\partial-x^2c\,\partial]\omega=0 \tag{16}$$

$$\omega=\beta^x/x_\nu x^\nu \qquad \beta c=0$$

where c^μ are arbitrary real constants; $x^2\equiv x_\nu x^\nu$, $x\,\partial=x_\nu(\partial/\partial x_\nu)$, $cx=c^\nu x_\nu$. After the substitution of the expression (15) into equation (2) with $k=\frac{1}{3}$ it implies that $\varphi(\omega)$ must satisfy the following system of nonlinear ordinary differential equations

$$\mathrm{d}\varphi/\mathrm{d}\omega=\mathrm{i}(\lambda/\beta_\nu\beta^\nu)(\bar{\varphi}\varphi)^{1/3}(\gamma\beta)\varphi$$

for which it is easy to obtain the general solution

$$\varphi=\exp[\mathrm{i}\lambda\varkappa(\gamma\beta)\omega]\chi\equiv[\cos(\lambda\varkappa\beta\omega)+\mathrm{i}(\gamma\beta/\beta)\sin(\lambda\varkappa\beta\omega)]\chi$$

and then (14).

It will be noted that the solution (14) is analytic in the coupling constant λ, in contrast to the solution obtained by Merwe (1981) with the help of the Heisenberg (1954) *ansatz*. Besides that

$$\bar{\psi}(x)\psi(x)=a/(x_\nu x^\nu)^3$$

i.e. $\bar{\psi}\psi$ dies off very fast when $x_\nu x^\nu\to\infty$. It is also noteworthy that such a solution is easy to generalise to the case of n spatial variables, the conformally invariant equation being

$$[\mathrm{i}\gamma\,\partial-\lambda(\bar{\psi}(x)\psi(x))^{1/n}]\psi(x)=0$$

and the solution takes the form

$$\psi(x)=\frac{\gamma x}{(x_\nu x^\nu)^{(n+1)/2}}\exp[\mathrm{i}\lambda\varkappa(\gamma\beta)\omega]\chi, \qquad \nu=0,1,\ldots,n$$

(here γ-matrices have appropriate structure (see e.g. Boerner 1970)). Using straightforward calculations one can make sure that the functions (18), stated below, satisfies equation (1) as well as equation (2) if $m=0$. This solution has been obtained by virtue of operator Q_{L} which is a linear combination of the Lorentz rotation generators

$$
\begin{aligned}
&Q_{\mathrm{L}}=\theta_a J_{0a} \qquad a=1,2,3\\
&J_{0a}=x_0\,\partial_a+x_a\,\partial_0-\tfrac{1}{2}\gamma_0\gamma_a
\end{aligned}
\tag{17}
$$

$$
\psi(x)=A(x)\varphi(\omega)
$$

$$
A(x)=\begin{pmatrix}
\frac{\theta_3}{\theta}s_+ & -\frac{\theta_-}{\theta}s_- & -\frac{\theta_3}{\theta}\frac{1}{s_+} & \frac{\theta_-}{\theta}\frac{1}{s_-}\\
\frac{\theta_+}{\theta}s_+ & \frac{\theta_3}{\theta}s_- & -\frac{\theta_+}{\theta}\frac{1}{s_+} & -\frac{\theta_3}{\theta}\frac{1}{s_-}\\
s_+ & 0 & \frac{1}{s_+} & 0\\
0 & s_- & 0 & \frac{1}{s_-}
\end{pmatrix}
\tag{18}
$$

$$
\varphi(\omega)=\begin{pmatrix}
\omega^{-1/2}[F_0\cos(\alpha+\alpha_0)+\mathrm{i}G_0\sin(\alpha+\alpha_0)]\\
\omega^{-1/2}[F_1\cos(\alpha+\alpha_1)+\mathrm{i}G_1\sin(\alpha+\alpha_1)]\\
-\quad[G_0\cos(\alpha+\alpha_0)+\mathrm{i}F_0\sin(\alpha+\alpha_0)]\\
-\quad[G_1\cos(\alpha+\alpha_1)+\mathrm{i}F_1\sin(\alpha+\alpha_1)]
\end{pmatrix}
$$

where: $\boldsymbol{\theta}=\{\theta_1,\theta_2,\theta_3\}$, α_0, α_1, F_0, F_1, G_0, G_1, $c=4(F_0G_0+F_1G_1)>0$ are arbitrary real constants:

$$
\begin{aligned}
&\omega=(\theta x_0)^2-(\boldsymbol{\theta}\cdot\boldsymbol{x})^2, \qquad s_\pm=(\theta x_0\pm\boldsymbol{\theta}\cdot\boldsymbol{x})^{1/2}\\
&\theta_\pm=\theta_1\pm\mathrm{i}\theta_2, \qquad \theta=(\theta_1^2+\theta_2^2+\theta_3^2)^{1/2}\\
&\alpha=\frac{\lambda c^k}{\theta(k-1)}\omega^{(1-k)/2}-\frac{m}{\theta}\sqrt{\omega}, \qquad k\neq 1;\\
&\alpha=-\frac{\lambda c}{2\theta}\ln\omega-\frac{m}{\theta}\sqrt{\omega}, \qquad k=1.
\end{aligned}
\tag{19}
$$

This solution is also analytic in the coupling constant λ and in the mass term, and

$$
\bar{\psi}(x)\psi(x)=\frac{c}{\sqrt{\omega}}=\frac{4(F_0G_0+F_1G_1)}{[(\theta x_0)^2-(\boldsymbol{\theta}\cdot\boldsymbol{x})^2]^{1/2}}
\tag{20}
$$

i.e. $\bar{\psi}\psi$ dies off when $x_\nu x^\nu\to\infty$.

The next solution has been obtained by means of the operator

$$
\begin{aligned}
&Q_{LD}=Q_{\mathrm{L}}+\varkappa D, \qquad \varkappa=\text{constant}\\
&D=x^\nu\,\partial_\nu-1/2k
\end{aligned}
\tag{21}
$$

which is admitted only by equation (2) and not by equation (1). We found an explicit solution in the case $k=1$ in such a form.

$$\psi(x) = A(x)\varphi(\omega)$$

$$A(x) = \begin{pmatrix} -\frac{\theta_3}{\theta} e^{\lambda_- s} & -\frac{\theta_-}{\theta} e^{\lambda_- s} & \frac{\theta_3}{\theta} e^{\lambda_+ s} & \frac{\theta_-}{\theta} e^{\lambda_+ s} \\ -\frac{\theta_+}{\theta} e^{\lambda_- s} & \frac{\theta_3}{\theta} e^{\lambda_- s} & \frac{\theta_+}{\theta} e^{\lambda_+ s} & -\frac{\theta_3}{\theta} e^{\lambda_+ s} \\ e^{\lambda_- s} & 0 & e^{\lambda_+ s} & 0 \\ 0 & e^{\lambda_- s} & 0 & e^{\lambda_+ s} \end{pmatrix} \tag{22}$$

$$\varphi(\omega) = \begin{pmatrix} G_0\omega^{i\beta} + F_0\omega^{-i\beta} \\ G_1\omega^{i\beta} + F_1\omega^{-i\beta} \\ \omega^{-1/2}\left(\frac{\theta+\varkappa}{\theta-\varkappa}\right)^{1/2}(G_0\omega^{i\beta} - F_0\omega^{-i\beta}) \\ \omega^{-1/2}\left(\frac{\theta+\varkappa}{\theta-\varkappa}\right)^{1/2}(G_1\omega^{i\beta} - F_1\omega^{-i\beta}) \end{pmatrix}$$

where: $\boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3\}$, $\varkappa$ are arbitrary real constants and F_0, F_1, G_0, G_1 are complex ones:

$$\theta_\pm = \theta_1 \pm i\theta_2, \qquad \theta = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2} \qquad \omega = (\theta x_0 - \boldsymbol{\theta}\cdot\boldsymbol{x})(\theta x_0 + \boldsymbol{\theta}\cdot\boldsymbol{x})^{(\theta-\varkappa)/(\theta+\varkappa)}$$

$$s = \ln(\theta x_0 + \boldsymbol{\theta x})/\theta + \varkappa, \qquad \beta = \frac{\lambda c_1}{2\theta}\left(\frac{\theta+\varkappa}{\theta-\varkappa}\right)^{1/2}, \qquad \lambda_\pm = \frac{-\varkappa \pm \theta}{2}, \qquad \theta > \varkappa \tag{23}$$

$$c_1 = 4\left(\frac{\theta+\varkappa}{\theta-\varkappa}\right)^{1/2}(F_0^*F_0 + F_1^*F_1 - G_0^*G_0 - G_1^*G_1).$$

This solution is also analytic in the coupling constant and

$$\bar{\psi}(x)\psi(x) = c_1/[(\theta x_0)^2 - (\boldsymbol{\theta}\cdot\boldsymbol{x})^2]^{1/2} \tag{24}$$

i.e. $\bar{\psi}\psi$ dies off as was previously the case (see (20)).

When $k \neq 1$ some particular exact solutions of equation (2) analogous to those given in (22) are provided by the *ansatz* (4) with $A(x)$ and $\varphi(\omega)$ having the form:

$$A(x) = \begin{pmatrix} -\frac{\theta_3}{\theta} e^{\mu_- s} & -\frac{\theta_-}{\theta} e^{\mu_- s} & \frac{\theta_3}{\theta} e^{\mu_+ s} & \frac{\theta_-}{\theta} e^{\mu_+ s} \\ -\frac{\theta_+}{\theta} e^{\mu_- s} & \frac{\theta_3}{\theta} e^{\mu_- s} & \frac{\theta_+}{\theta} e^{\mu_+ s} & -\frac{\theta_3}{\theta} e^{\mu_+ s} \\ e^{\mu_- s} & 0 & e^{\mu_+ s} & 0 \\ 0 & e^{\mu_- s} & 0 & e^{\mu_+ s} \end{pmatrix} \tag{25}$$

$$\varphi(\omega) = \begin{pmatrix} \varphi_0(\omega) \\ \varphi_1(\omega) \\ \omega^{\mu_+/(\varkappa-\theta)}\varphi_2(\omega) \\ \omega^{\mu_+/(\varkappa-\theta)}\varphi_3(\omega) \end{pmatrix}$$

where $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ are defined from the following system of ordinary differential equations

$$
\begin{aligned}
&\varphi_0^*\varphi_2+\varphi_0\varphi_2^*+\varphi_1^*\varphi_3+\varphi_1\varphi_3^*=-\tfrac{1}{2}c_2=\text{constant}\\
&\frac{d\varphi_0}{d\omega}=\frac{i\lambda}{2\theta}c_2^k\omega^{\mu_+(k+1)/(\varkappa-\theta)}\varphi_2(\omega)\\
&\frac{d\varphi_2}{d\omega}=\frac{i\lambda}{2\theta}c_2^k\omega^{[\mu_+(k-1)/(\varkappa-\theta)]-1}\frac{\theta+\varkappa}{\theta-\varkappa}\varphi_0(\omega)\\
&\frac{d\varphi_1}{d\omega}=\frac{i\lambda}{2\theta}c_2^k\omega^{\mu_+(k+1)/(\varkappa-\theta)}\varphi_3(\omega)\\
&\frac{d\varphi_3}{d\omega}=\frac{i\lambda}{2\theta}\frac{\theta+\varkappa}{\theta-\varkappa}c_2^k\omega^{[\mu_+(k-1)/(\varkappa-\theta)]-1}\varphi_1(\omega)
\end{aligned}
\tag{26}
$$

$\mu_\pm=(-\varkappa\pm\theta k)/2k$, c_2 is an arbitrary real constant, and $s, \omega, \theta_\pm$ and $\theta_3\theta$ are defined in (23).

Below we present the explicit form of transformations admitted by equations (1) or (2). They can be used to generate new exact solutions of the equations in accordance with the formula (10).

The conformal transformations

$$
\begin{aligned}
&x'_\mu=(x_\mu-c_\mu x^2)/\sigma(x) \qquad \sigma(x)\equiv 1-2cx+c^2x^2\\
&\psi'(x')=R_{\text{conf}}\psi(x)=\sigma(x)(1-\gamma c\gamma x)\psi(x)\\
&R_{\text{conf}}^{-1}=\sigma^{-2}(x)(1-\gamma x\gamma c).
\end{aligned}
\tag{27}
$$

The transformation of dilatation

$$
x'_\mu=e^\alpha x_\mu, \qquad \psi'(x')=R_D\psi(x)=e^{-\alpha/2k}\psi(x) \qquad R_D^{-1}=e^{\alpha/2k}. \tag{28}
$$

The transformations of rotations

$$
\begin{aligned}
&x'_0=x_0, \qquad \boldsymbol{x}'=\boldsymbol{x}\cos\delta+\frac{\boldsymbol{x}\times\boldsymbol{\delta}}{\delta}\sin\delta+\frac{\boldsymbol{\delta}(\boldsymbol{x}\cdot\boldsymbol{\delta})}{\delta^2}(1-\cos\delta);\\
&\psi'(x')=R_{\text{rot}}\psi(x)=(\cos\tfrac{1}{2}\delta+(i/\delta)(\boldsymbol{\Sigma}\cdot\boldsymbol{\delta})\sin\tfrac{1}{2}\delta)\psi(x)\\
&R_{\text{rot}}^{-1}=\cos\tfrac{1}{2}\delta-(i/\delta)(\boldsymbol{\Sigma}\cdot\boldsymbol{\delta})\sin\tfrac{1}{2}\delta
\end{aligned}
\tag{29}
$$

where $\boldsymbol{\delta}=(\delta_1, \delta_2, \delta_3)$, $\delta=(\delta_1^2+\delta_2^2+\delta_3^2)^{1/2}$, $\boldsymbol{\Sigma}=\sigma_0\times\boldsymbol{\sigma}$, $\boldsymbol{\sigma}=\{\sigma_1, \sigma_2, \sigma_3\}$ are Pauli matrices, σ_0 is the identity 2×2 matrix.

The Lorentz transformations

$$
\begin{aligned}
&x'_0=x_0\cosh\theta_1-x_1\sinh\theta_1\\
&x'_1=x_1\cosh\theta_1-x_0\sinh\theta_1\\
&\psi'(x')=R_{L_1}\psi(x)=(\cosh\tfrac{1}{2}\theta_1+\gamma_0\gamma_1\sinh\tfrac{1}{2}\theta_1)\psi(x)\\
&R_{L_1}^{-1}=\cosh\tfrac{1}{2}\theta_1-\gamma_0\gamma_1\sinh\tfrac{1}{2}\theta_1.
\end{aligned}
\tag{30}
$$

The rest of the Lorentz transformations are analogous to those given above.

The transformations of displacements

$$x'_\mu = x_\mu + a_\mu, \qquad \psi'(x') = \psi(x). \tag{31}$$

In formulae (26)–(31) c^μ, α, δ_a, θ_a, a_μ are arbitrary real constants.

In conclusion we would like to give a simple example of the transformation of the well known plane-wave solution of the free massless Dirac equation into the new solution using formulae (10) and (27).

$$\psi_{\mathrm{pw}}(x) = \exp(\mathrm{i}kx)\chi, \qquad k^2 \equiv k_\mu k^\mu = 0,$$

χ is a space–time independent spinor, $\bar{\chi}\chi$ = constant:

$$\psi_{\mathrm{pw}} \to \Psi(x) = \frac{1 - \gamma x \gamma c}{\sigma^2(x)} \exp\{\mathrm{i}k^\mu [(x_\mu - c_\mu x^2)/\sigma(x)]\}\chi,$$

$$k_\mu k^\mu = 0.$$

It is easy to verify that it is a solution of the free massless Dirac equation but it is no longer a plane-wave solution because of the nonlinear character of the conformal transformations. Moreover,

$$\bar{\psi}(x)\psi(x) = \bar{\chi}\chi/\sigma^3(x) \equiv \text{constant}/(1 - 2cx + c^2x^2)^3$$

and dies off very fast when $x^2 \equiv x_\nu x^\nu \to \infty$ while

$$\bar{\psi}_{\mathrm{pw}}(x)\psi_{\mathrm{pw}}(x) = \bar{\chi}\chi = \text{constant}.$$

Acknowledgment

We would like to express our gratitude to the referees for their comments and useful suggestions.

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